MATHEMATICS

Introduction to Series

Infinite Series: - Let $\{u_n\}$ be a sequence of real numbers.

Let $s_1=u_1$

$s_2=u_1+u_2$

$s_3=u_1+u_2+u_3$

……………

$s_n=u_1+u_2+u_3+\text{------}+u_n$

…………..

then the sequence $\{s_n\}$ is called an infinite series. The number u_n is called the n^{th} term of the series. The number s_n is called the n^{th} partial sum of the series. The infinite series $\{s_n\}$ is denoted by

$\sum_{n=1}^{\infty} u_n$ (or) $u_1 + u_2 + u_3 + \ldots\ldots$ (or) $\sum u_n$.

Some times the infinite series begins with u_0. In this case

$s_1=u_0+u_1$

$s_2= u_0+u_1+u_2$

$s_3= u_0+u_1+u_2+u_3$

…………

$s_n= u_0+u_1+u_2+u_3+\text{------}+u_n$

…………

In this case we write the series as $\sum_{n=0}^{\infty} u_n$

If the sequence $\{s_n\}$ converges to l then we say that $\sum u_n$ converges to l.

The number l is called the sum of the series and we write $\sum_{n=0}^{\infty} u_n = l$. If the sequence $\{s_n\}$ diverges then we say that $\sum u_n$ diverges.

Results

1. If $\sum u_n$ converges to A, $\sum v_n$ converges to B then $\sum (u_n + v_n)$ converges to A+B
2. If $\sum u_n$ converges to A, k∈R then $\sum ku_n$ converges to kA
3. If $\sum u_n$ diverges and k∈R, k≠0 then $\sum ku_n$ diverges to ∞
4. If $\sum u_n$ and $\sum v_n$ diverges then $\sum (u_n + v_n)$ diverges
5. If $\sum u_n$ converges and $\sum v_n$ diverges then $\sum (u_n + v_n)$ diverges
6. If $\sum u_n$ converges then $\underset{n\to\infty}{\text{lt}} u_n = 0$

The converse of the above result need not be true i.e., if $\underset{n\to\infty}{\text{lt}} u_n = 0$ then $\sum u_n$ may or may not converge.

Note: If $\underset{n\to\infty}{\text{lt}} u_n \neq 0$ then $\sum u_n$ is divergent

Problems

1. Prove that $\sum \frac{n+1}{n+2}$ is divergent.

A. Here $u_n = \frac{n+1}{n+2}$

$$\underset{n\to\infty}{\text{lt}} u_n = \underset{n\to\infty}{\text{lt}} \frac{n+1}{n+2} = \underset{n\to\infty}{\text{lt}} \frac{1+1/n}{1+2/n} = 1 \neq 0$$

$\therefore \sum u_n$ is divergent.

2. Prove that $\sum \frac{n}{2n+1}$ is divergent.

A. Here $u_n = \frac{n}{2n+1}$

$$\underset{n\to\infty}{\text{lt}} u_n = \underset{n\to\infty}{\text{lt}} \frac{n}{2n+1} = \underset{n\to\infty}{\text{lt}} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$$

$\therefore \sum u_n$ is divergent

Series of non negative terms:-

If $\sum u_n$ is a series of non negative terms then $u_n \geq 0, \forall n$

If $\sum u_n$ is a series of positive terms then $u_n > 0, \forall n$

1) Geometric series:- The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + --------$ is called geometric series.

This series converges if $0 < r < 1$ and diverges if $r \geq 1$.

2) Auxiliary series (or) p-series:- The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + --------$

($p \in R$) is called auxiliary series.

This series is convergent if $p>1$ and divergent if $p \leq 1$

Eg:- (1) $\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent. Here $r = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$

(2) $\sum_{n=0}^{\infty} 3^n$ is divergent. Here r=3 and 3>1

(3) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Here p=2>1

(4) $\sum_{n=1}^{\infty} n^2$ is divergent. Here p=-2<1

(5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ is divergent. Here p= $\frac{1}{2}$ <1

(6) $\sum \frac{1}{n}$ is divergent. Here p=1

$\sum_{n=0}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots$ is called harmonic series.

(7) $\sum_{n=1}^{\infty} 1 = \sum \frac{1}{n^0}$ is divergent. Here p=0

(8) $\sum_{n=1}^{\infty} \frac{1}{n.n^{\frac{1}{100}}} = \sum_{n=0}^{\infty} \frac{1}{n^{\frac{101}{100}}}$ is convergent. Here p>1

Comparison test (1st type):-

Theorem (1):- If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that

(i) ∃ a +ve integer m and $k \in R^+$ such that $u_n \leq kv_n, \forall n \geq m$ (ii) $\sum v_n$ is convergent then $\sum u_n$ is also convergent.

Theorem (2):- If $\sum u_n$ and $\sum v_n$ are two series

(i) ∃ a +ve integer m and $k \in R^+$ such that $u_n \geq kv_n, \forall n \geq m$ (ii) $\sum v_n$ is divergent then $\sum u_n$ is also divergent.

Problems

1. Test the convergence of $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

A. Let $u_n = \frac{1}{n^2+1}$

We know that $n^2+1>n^2$, ∀ n≥1

$$\Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}$$

$$\Rightarrow u_n < v_n \forall n \geq 1 \text{ where } v_n = \frac{1}{n^2}$$

Now $\sum v_n = \sum \frac{1}{n^2}$ is convergent

∴ By comparison test first type, $\sum u_n$ is convergent.

2. Test the convergence of $\sum \frac{1}{2n^3-1}$

A. Let $u_n = \frac{1}{2n^3 - 1}$

We know that $2n^3 > n^3$, $\forall n \geq 1$

$$\Rightarrow \quad 2n^3 - 1 \geq n^3$$

$$\Rightarrow \quad \frac{1}{2n^3 - 1} \leq \frac{1}{n^3}$$

$$\Rightarrow \quad u_n < v_n \forall n \geq 1 \text{ where } v_n = \frac{1}{n^3}$$

Now $\sum v_n = \sum \frac{1}{n^3}$ is convergent

$\therefore$ By comparison test first type, $\sum u_n$ is convergent.